

Local estimates for elliptic equations arising in conformal geometry

Yan He Weimin Sheng

Abstract

In this paper we consider Yamabe type problem for higher order curvatures on manifolds with totally geodesic boundaries. We prove local gradient and second derivative estimates for solutions to the fully nonlinear elliptic equations associated with the problems.

1 Introduction

Let (M^n, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$. The Schouten tensor of g is defined by

$$A_g = \frac{1}{n-2} \left(\text{Ric}_g - \frac{R_g}{2(n-1)} g \right),$$

where Ric_g and R_g are the Ricci and scalar curvatures of g , respectively. The k -curvature (or σ_k curvature) is defined to be the k -th elementary symmetric function σ_k of the eigenvalues $\lambda(g^{-1}A_g)$ of $g^{-1}A_g$. If $\tilde{g} = e^{-2u}g$ is a metric conformal to g , the Schouten tensor transforms according to the formula

$$A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g,$$

where ∇u and $\nabla^2 u$ denote the gradient and Hessian of u with respect to g . Consequently, the problem of conformally deforming a given metric to one with prescribed σ_k -curvature reduces to solving the partial differential equation

$$\sigma_k \left(\lambda \left(g^{-1} \left[\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g \right] \right) \right) = \psi(x) e^{-2ku}. \quad (1.1)$$

For compact manifolds without boundary, the existence of the solutions to the equation (1.1) has been studied by many authors (see [CGY1, CGY2, GW2, GW3, LL1,

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LL2, GV1, GV2, TW1, TW2, STW, GeW, V2] etc.) since these equations were first introduced by J. A. Viaclovsky [V1]. C^1 and C^2 estimates have also been studied extensively, see [Cn1, GW1, GW2, LL1, STW, W2] for local interior estimates and [V2] for global estimates.

Another interesting problem is to study the fully nonlinear equation (1.1) on a compact Riemannian manifold (M^n, g) with boundary ∂M . In [G], Bo Guan studied the existence problem under the Dirichlet boundary condition. There are many pioneering works on the Dirichlet problems for fully nonlinear elliptic equations, see [CNS, Tr2] etc.. The Neumann problem for (1.1) has been studied by S. Chen [Cn2, Cn3], Jin-Li-Li [JLL], Jin [J] and Li-Li [LL3], etc.. Under various conditions, they derive local estimates for solutions and establish some existence results. Before introducing the problem, we need the following definitions.

Define

$$\Gamma_k = \{\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, 1 \leq j \leq k\},$$

and $1 \leq k \leq n$, where σ_k is the k -th elementary symmetric function defined by

$$\sigma_k(\Lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

for all $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$. We also denote $\sigma_0 = 1$. Therefore we have the relation $\Gamma_n \subset \Gamma_{n-1} \subset \dots \subset \Gamma_1$. For a 2-symmetric form S defined on (M^n, g) , $S \in \Gamma_k$ means that the eigenvalues of S , $\lambda(g^{-1}S)$ lie in Γ_k . We also denote $\Gamma_k^- = -\Gamma_k$.

Let (M^n, g) , $n \geq 3$, be a smooth compact Riemannian manifold with nonempty smooth boundary ∂M . We denote the mean curvature and the second fundamental form of ∂M by h_g and $L_{\alpha\beta}$, where $\{x^\alpha\}_{1 \leq \alpha \leq n-1}$ is the local coordinates on the boundary ∂M , and $\frac{\partial}{\partial x^n}$ the unit inner normal with respect to the metric. In this paper similar as [E2, Cn2] we use Fermi coordinates in a boundary neighborhood. In these local coordinates, we take the geodesic in the inner normal direction $\nu = \frac{\partial}{\partial x^n}$ parameterized by arc length, and (x^1, \dots, x^{n-1}) forms a local chart on the boundary. The metric can be expressed as $g = g_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2$. The Greek letters $\alpha, \beta, \gamma, \dots$ stand for the tangential direction indices, $1 \leq \alpha, \beta, \gamma, \dots \leq n-1$, while the Latin letters i, j, k, \dots stand for the full indices, $1 \leq i, j, k, \dots \leq n$. In Fermi coordinates, the half ball is defined by $\overline{B}_r^+ = \{x_n \geq 0, \sum_i x_i^2 \leq r^2\}$ and the segment on the boundary by $\Sigma_r = \{x_n = 0, \sum_i x_i^2 \leq r^2\}$. Under the conformal change of the metric $\tilde{g} = e^{-2u} g$, the second fundamental form satisfies

$$\tilde{L}_{\alpha\beta} e^u = \frac{\partial u}{\partial \nu} g_{\alpha\beta} + L_{\alpha\beta}.$$

The boundary is called umbilic if the second fundamental form $L_{\alpha\beta} = \tau_g g_{\alpha\beta}$, where τ_g is the function defined on ∂M . A totally geodesic boundary is umbilic with $\tau_g \equiv 0$. Note that the umbilicity is conformally invariant. When the boundary is umbilic, the above formula becomes

$$\tau_{\tilde{g}} e^{-u} = \frac{\partial u}{\partial \nu} + \tau_g.$$

The k -Yamabe problem with umbilic boundary becomes to considering the following equation:

$$\begin{cases} \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g \right] \right) \right) = e^{-2u} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = \tau_{\tilde{g}} e^{-u} - \tau_g & \text{on } \partial M. \end{cases} \quad (1.2)$$

In [Cn2, Cn3], [JLL] and [J], the authors established the a priori estimates and obtained some existence results for (1.2).

In this paper, we will generalize their results to more general equations, which in particular include the equation (1.2). In [GV3], Gursky and Viaclovsky introduced a modified Schouten tensor

$$A_g^t = \frac{1}{n-2} \left(\text{Ric}_g - \frac{tR_g}{2(n-1)} g \right),$$

where $t \in \mathbb{R}$ is a parameter. When $t = 1$, A_g^1 is just the Schouten tensor; $t = n-1$, A_g^{n-1} is the Einstein tensor; while $t = 0$, A_g^0 is the Ricci tensor. This tensor A_g^t is in fact a constant multiple of the tensor $sA_g + \frac{(1-s)}{2(n-1)} R_g g$ which is introduced in [LL1], i.e. $A_g^{\frac{2s-1}{s}} = \frac{1}{s} (sA_g + \frac{(1-s)}{2(n-1)} R_g g)$. Under the conformal change of the metric $\tilde{g} = e^{-2u} g$, $A_{\tilde{g}}^t$ satisfies

$$A_{\tilde{g}}^t = A_g^t + \nabla^2 u + \frac{1-t}{n-2} (\Delta u) g + du \otimes du - \frac{2-t}{2} |\nabla u|^2 g.$$

In [LS] and [SZ], we have studied

$$\sigma_k \left(\lambda \left(g^{-1} \left[A_g^t + \nabla^2 u + \frac{1-t}{n-2} (\Delta u) g + du \otimes du - \frac{2-t}{2} |\nabla u|^2 g \right] \right) \right) = f(x) e^{-2ku}$$

for $t \leq 1$ or $t \geq n-1$. By use of the parabolic approach, we obtained some existence results. Let (M, g) be a compact, connected Riemannian manifold of dimension $n \geq 3$ with umbilic boundary ∂M , W be a $(0, 2)$ symmetric tensor on (M^n, g) . Motivated by [Cn1], in this paper we study the following equation

$$\begin{cases} F(g^{-1}W) = f(x, u) & \text{in } M \\ \frac{\partial u}{\partial \nu} = \tilde{\tau} e^{-u} - \tau & \text{on } \partial M \end{cases} \quad (1.3)$$

where F satisfies some fundamental structure conditions listed below, and τ is the principal curvature of the boundary ∂M . We will establish local a priori estimates for the solutions to the equation (1.3). After that, we will give some applications. More applications, see [HS1, HS2].

We now describe the fundamental structure conditions for F .

Let Γ be an open convex cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$. Suppose that $F(\lambda)$ is a homogeneous symmetric function of degree one in Γ normalized with $F(e) = F((1, \dots, 1)) = 1$. Moreover, F satisfies the following in Γ :

- (A1) F is positive.
- (A2) F is concave (i.e., $\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}$ is negative semi definite).
- (A3) F is monotone (i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive).
- (A4) $\frac{\partial F}{\partial \lambda_i} \geq \varepsilon \frac{F}{\sigma_1}$, for some constant $\varepsilon > 0$, for all i .

The conditions (A1), (A2), (A3) and (A4) are similar as those in [Cn2]. Before stating the theorems, we introduce the following notations. Let $f(x, z) : M^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a given positive function. Let $u = u(x) : M^n \rightarrow \mathbb{R}$ be a solution to (1.3). We define

$$\overline{c_{sup}}(r) = \sup_{\overline{B}_r^+} (f + |\nabla_x f(x, u)| + |f_z(x, u)| + |\nabla_x^2 f(x, u)| + |\nabla_x f_z(x, u)| + |f_{zz}(x, u)|)$$

or

$$c_{sup}(r) = \sup_{B_r} (f + |\nabla_x f(x, u)| + |f_z(x, u)| + |\nabla_x^2 f(x, u)| + |\nabla_x f_z(x, u)| + |f_{zz}(x, u)|),$$

which varies with boundary or interior estimates.

Now we turn to the first equation: let

$$W = \nabla^2 u + \frac{1-t}{n-2} (\Delta u) g + a(x) du \otimes du + b(x) |\nabla u|^2 g + S, \quad (1.4)$$

where t is a constant satisfying $t \leq 1$, S a 2-symmetric form defined on M , and $a(x)$, $b(x)$ are two smooth functions on M . The derivatives are covariant with respect to the metric g . We have

Theorem 1. *Let F satisfy the structure conditions (A1)-(A4) in a corresponding cone Γ , $a(x) = a$, $b(x) = b$ are two constants, $S = A$ the Schouten tensor. Suppose that the boundary ∂M is totally geodesic. Let $u(x)$ be a C^4 solution to the equation*

$$\begin{cases} F(g^{-1}(\nabla^2 u + \frac{1-t}{n-2} \Delta u) g + a(x) du \otimes du + b(x) |\nabla u|^2 g + S) = f(x, u) & \text{in } \overline{B}_r^+, \\ \frac{\partial u}{\partial x^n} = 0 & \text{on } \Sigma_r, \end{cases} \quad (1.5)$$

and $W \in \Gamma$. Suppose that $|\nabla f| \leq \Lambda f$, $|f_z| \leq \Lambda f$ for some constant $\Lambda > 0$. If $\frac{1-t}{n-2}a - b \geq \delta_1 > 0$, $a + nb \leq -\delta_3 < 0$, and $a \geq 0$, then

$$\sup_{\overline{B}_r^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where C depends on $r, n, \varepsilon, \Lambda, \delta_1, \delta_3, a, b, \|A\|_{C^2(\overline{B}_r^+)}, \|g\|_{C^3(\overline{B}_r^+)} \text{ and } \overline{c_{sup}}(r)$.

When $t = 1$, and $a = 1, b = -\frac{1}{2}$, the boundary estimates have been obtained by S. Chen [Cn2, Cn3], Jin-Li-Li [JLL] and Jin [J] for some special cases. When $t = 1$, the local interior estimates have been discussed by S. Chen in [Cn1] for general functions $a(x), b(x)$ and a general 2-symmetric tensor S . We just focus on the interior estimates for the same equation, we may get

Theorem 2. *Let F satisfy the structure conditions (A1)-(A4) in a corresponding cone Γ . Let $u(x)$ be a C^4 solution to the equation*

$$F(g^{-1}(\nabla^2 u + \frac{1-t}{n-2} \Delta u g + a(x)du \otimes du + b(x)|\nabla u|^2 g + S)) = f(x, u) \quad (1.6)$$

in a local geodesic ball $B_r \subset M$ and $W \in \Gamma$. Suppose that $|\nabla f| \leq \Lambda f$, $|f_z| \leq \Lambda f$ for some constant $\Lambda > 0$.

Case (a). If $\Gamma \subset \Gamma_2^+$, $\frac{1-t}{n-2}a(x) - b(x) \geq \delta_1 > 0$ and $\min\{2ab + b^2, b^2\} \geq \delta_2 > 0$, then

$$\sup_{\overline{B}_r^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where C depends only on $r, n, \Lambda, \delta_1, \delta_2, \|a\|_{C^2(B_r)}, \|b\|_{C^2(B_r)}, \|S\|_{C^2(B_r)}, \|g\|_{C^3(B_r)} \text{ and } c_{sup}(r)$.

Case (b). If $\frac{1-t}{n-2}a(x) - b(x) \geq \delta_1 > 0$, $a(x) + nb(x) \leq -\delta_3 < 0$ and $a(x) \geq 0$, then we have

$$\sup_{\overline{B}_r^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where C depends on $r, n, \varepsilon, \Lambda, \delta_1, \delta_3, \|a\|_{C^2(B_r)}, \|b\|_{C^2(B_r)}, \|S\|_{C^2(B_r)}, \|g\|_{C^3(B_r)} \text{ and } c_{sup}(r)$.

Remark. In case (a), the condition $\min\{2ab + b^2, b^2\} \geq \delta_2 > 0$ may be replaced by $\min\{b^2 + 2ab - 2n\|a\|_\infty b - a^2, b^2\} \geq \delta_2 > 0$. The proof can be found in the proof of Theorem 2, case (a). The later condition is better than the former when $a > 0$.

The a priori estimates in Theorem 1 and Theorem 2 rely on the signs of $a(x)$ and $b(x)$. In fact, in [STW] the authors give a counterexample to show that there is no regularity if $a(x) = 0$ and $b(x) > 0$ when $t = 1$. It is well known that the equation

(1.6) has another elliptic branch, namely when the eigenvalues λ lie in the negative cone Γ_k^- . Now we consider the second equation. Let

$$V = \frac{t-1}{n-2}(\Delta u)g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2g + S,$$

where t is a constant satisfying $t \geq n-1$. We have

Theorem 3. *Let F satisfy the structure conditions (A1)-(A4), $a(x) = a, b(x) = b$ are two constants, $-S = A$ the Schouten tensor. Suppose that the boundary ∂M is totally geodesic. Let $u(x)$ be a C^4 solution to the equation*

$$\begin{cases} F(g^{-1}(\frac{t-1}{n-2}(\Delta u)g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2g + S)) = f(x, u) & \text{in } \overline{B}_r^+, \\ \frac{\partial u}{\partial x^n} = 0 & \text{on } \Sigma_r, \end{cases} \quad (1.7)$$

and $V \in \Gamma$, $t > n-1$. Suppose that $|\nabla f| \leq \Lambda f$, $|f_z| \leq \Lambda f$ for some constant $\Lambda > 0$. If $a + nb \geq \delta_3 > 0, a \geq 0$, then

$$\sup_{\overline{B}_{\frac{r}{2}}^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where C depends on $r, n, \varepsilon, \Lambda, \delta_3, a, b, \|A\|_{C^2(\overline{B}_r^+)}, \|g\|_{C^3(\overline{B}_r^+)} \text{ and } \overline{c_{sup}}(r)$.

Similar with Theorem 2, if we just focus on the interior estimates for the same equation, we can get the following theorem for general functions $a(x)$, $b(x)$ and general 2-symmetric tensor S .

Theorem 4. *Let F satisfy the structure conditions (A1)-(A4) in a corresponding cone Γ . Let $u(x)$ be a C^4 solution to the equation*

$$F(g^{-1}(\frac{t-1}{n-2}(\Delta u)g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2g + S)) = f(x, u) \quad (1.8)$$

in a local geodesic ball $B_r \subset M$ and $V \in \Gamma$. Suppose that $|\nabla f| \leq \Lambda f$, $|f_z| \leq \Lambda f$ for some constant $\Lambda > 0$.

Case (a). If $\Gamma \subset \Gamma_2^+$, $\frac{t-1}{n-2}a(x) + b(x) \geq \delta_1 > 0$, $\min\{2ab + b^2, b^2\} \geq \delta_2 > 0$, then

$$\sup_{B_{\frac{r}{2}}} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where C depends only on $r, n, \Lambda, \delta_1, \delta_2, \|a\|_{C^2(B_r)}, \|b\|_{C^2(B_r)}, \|S\|_{C^2(B_r)}, \|g\|_{C^3(B_r)} \text{ and } c_{sup}(r)$.

Case (b). If $a(x) + nb(x) \geq \delta_3 > 0, a(x) \geq 0$, then

$$\sup_{B_{\frac{r}{2}}} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where C depends on $r, n, \Lambda, \delta_3, \|a\|_{C^2(B_r)}, \|b\|_{C^2(B_r)}, \|S\|_{C^2(B_r)}, \|g\|_{C^3(B_r)} \text{ and } c_{sup}(r)$.

Our idea of proof is from [Cn1, Cn2, Cn3], that is we estimate the quantity $K := \Delta u + a(x)|\nabla u|^2$ rather than estimate the gradient and second derivatives separately. This idea was first used by Sophie Chen in [Cn1]. As in [Cn2, Cn3] we show that the function Ke^{px_n} does not attain its maximum on the boundary, where x_n is the distance to the boundary. The main point in our argument is the observation that there exists a suitable conformal transformation such that the metric has some nice geometric properties on the boundary (Lemma 5). We would like to mention a different method in getting the boundary estimates [JLL] and [J]. For the Neumann problem of the Monge-Ampère equation, the estimates were first obtained in [LTU].

This paper is organized as follows. We begin with some background in Section 2. In Section 3, we discuss the applications which are based on the a priori estimates in Theorem 1 to get the existence result of k -Yamabe problem. In Section 4 and Section 5 we first prove Theorem 2 and Theorem 4 respectively. We then prove the maximum of K does not appear on the boundary, therefore Theorem 1 and Theorem 3 can be concluded by the similar arguments of the case (b) of Theorems 2 and 4 respectively.

2 Preliminaries

In this section, we give some basic facts about homogeneous symmetric functions and show some outcomes by direct calculation under Fermi coordinates. All of the facts can be found in the literatures cited below.

From Lemma 1 and Lemma 2 below, we can conclude that F satisfies (A1)-(A4).

Lemma 1. (*[U]*) *Let Γ be an open convex cone with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma$, and let $e = (1, \dots, 1)$ be the identity. Suppose that F is a homogeneous symmetric function of degree one normalized with $F(e) = 1$, and that F is concave in Γ . Then*

- (a) $\sum_i \lambda_i \frac{\partial F}{\partial \lambda_i} = F(\lambda)$, for $\lambda \in \Gamma$;
- (b) $\sum_i \frac{\partial F}{\partial \lambda_i} \geq F(e) = 1$, for $\lambda \in \Gamma$.

Lemma 2. (*[Tr2, LT]*) *Let $G = \left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}}$, $0 \leq l < k \leq n$. Then*

- (a) *G is positive and concave in Γ_k ;*
- (b) *G is monotone in Γ_k , i.e., the matrix $G^{ij} = \frac{\partial G}{\partial W_{ij}}$ is positive definite;*
- (c) *Suppose $\lambda \in \Gamma_k$. For $0 \leq l < k \leq n$, the following is the Newton-Maclaurin inequality*

$$k(n-l+1)\sigma_{l-1}\sigma_k \leq l(n-k+1)\sigma_l\sigma_{k-1}.$$

The following two lemmas will be used in proving Theorem 1 and 2. Let us review some formulas on the boundary under Fermi coordinates (see [E2] or [Cn3]). The metric is expressed as $g = g_{\alpha\beta}dx^\alpha dx^\beta + (dx^n)^2$. The Christoffel symbols satisfy

$$\Gamma_{\alpha\beta}^n = L_{\alpha\beta}, \Gamma_{\alpha n}^\beta = -L_{\alpha\gamma}g^{\gamma\beta}, \Gamma_{\alpha n}^n = 0, \Gamma_{\alpha\beta}^\gamma = \tilde{\Gamma}_{\alpha\beta}^\gamma,$$

on the boundary, where we denote the tensors and covariant derivations with respect to the induced metric on the boundary by a tilde (e.g. $\tilde{\Gamma}_{\alpha\beta}^\gamma, \tau_{\tilde{\alpha}\tilde{\beta}}$). When the boundary is umbilic, we have

$$\Gamma_{\alpha\beta}^n = \tau g_{\alpha\beta}, \Gamma_{\alpha n}^\beta = -\tau \delta_{\alpha\beta}, \Gamma_{\alpha n}^n = 0.$$

Lemma 3. (see [Cn3]) Suppose boundary ∂M is umbilic. Let u satisfy $u_n := \frac{\partial u}{\partial x^n} = -\tau + \tilde{\tau}e^{-u}$, where $\tilde{\tau}$ is constant. Then on the boundary we have

$$u_{n\alpha} = -\tau_\alpha + \tau u_\alpha - \tilde{\tau} u_\alpha e^{-u}; \quad (2.1)$$

and

$$\begin{aligned} u_{\alpha\beta n} = & (2\tau - \tilde{\tau}e^{-u})u_{\alpha\beta} - \tau u_{nn}g_{\alpha\beta} + \tilde{\tau}u_\alpha u_\beta e^{-u} - \tau_{\tilde{\alpha}\tilde{\beta}} + \tau_\alpha u_\beta + \tau_\beta u_\alpha \\ & - \tau_\gamma u_\gamma g_{\alpha\beta} + R_{n\beta\alpha n}(-\tau + \tilde{\tau}e^{-u}) - \tau(-\tau + \tilde{\tau}e^{-u})^2 g_{\alpha\beta}. \end{aligned} \quad (2.2)$$

Lemma 4. Suppose the boundary ∂M is totally geodesic and $u_n = 0$ on the boundary. Then we have on the boundary

$$W_{\alpha\beta n} = \frac{1-t}{n-2}u_{nnn}g_{\alpha\beta} + a_n u_\alpha u_\beta + b_n (\Sigma_\gamma u_\gamma^2) g_{\alpha\beta} + S_{\alpha\beta n}, \quad (2.3)$$

$$V_{\alpha\beta n} = \frac{t-1}{n-2}u_{nnn}g_{\alpha\beta} - a_n u_\alpha u_\beta - b_n (\Sigma_\gamma u_\gamma^2) g_{\alpha\beta} + S_{\alpha\beta n}. \quad (2.4)$$

Proof. By the boundary condition we know that $\tilde{\tau} = \tau = 0$. From formulas (2.1) and (2.2) we have $u_{n\alpha} = 0$ and $u_{\alpha\beta n} = 0$. Then

$$\begin{aligned} W_{\alpha\beta n} = & u_{\alpha\beta n} + \frac{1-t}{n-2}\Sigma_k u_{kkn}g_{\alpha\beta} + a u_{\alpha n}u_\beta + a u_\alpha u_{\beta n} + a_n u_\alpha u_\beta \\ & + 2b \Sigma_k u_{kn}u_k g_{\alpha\beta} + b_n (\Sigma_k u_k^2) g_{\alpha\beta} + S_{\alpha\beta n} \\ = & \frac{1-t}{n-2}u_{nnn}g_{\alpha\beta} + a_n u_\alpha u_\beta + b_n (\Sigma_\gamma u_\gamma^2) g_{\alpha\beta} + S_{\alpha\beta n}. \end{aligned}$$

For V_{ij} we can get the equalities in the same way.

□

Lemma 5. *Let (M^n, g) be a compact Riemannian manifold with boundary and dimensional $n \geq 3$. Assume that the boundary ∂M is totally geodesic. Then at any boundary point $P \in \partial M$, there exists a conformal metric $\bar{g} = e^{-2\bar{u}}g$ such that (i) $\bar{u}_n = 0$ on ∂M and the boundary ∂M is still totally geodesic, (ii) $\bar{R}_{ij}(P) = 0$ for $1 \leq i, j \leq n$, (iii) $\bar{R}_{nn,n}(P) = 0, \bar{R}_{\alpha n, \beta}(P) = 0, 1 \leq \alpha, \beta \leq n-1$, and (iv) $\bar{R}_{\alpha \beta, n}(P) = 0, 1 \leq \alpha, \beta \leq n-1$.*

Proof. As the proof of Lemma 3.3 in [E2], consider the first eigenvalue $\lambda_1(L)$ of the conformal Laplacian with respect to the boundary condition

$$\begin{cases} L\varphi + \lambda_1(L)\varphi = 0 & \text{on } M, \\ \frac{\partial}{\partial x^n}\varphi + \frac{n-2}{2}h\varphi = 0 & \text{on } \partial M, \end{cases} \quad (2.5)$$

where $L = \Delta - \frac{n-2}{4(n-1)}R$, R is the scalar curvature, h is the mean curvature of the boundary and $\frac{\partial}{\partial x^n}$ is the inward norm derivative with respect to the metric g . Since ∂M is totally geodesic, $h = 0$. Let φ_1 be the first eigenfunction for the conformal Laplacian with respect to the boundary condition (2.5), then $\varphi_1 > 0$. Set $g_1 = \varphi_1^{\frac{4}{n-2}}g$. The transformation law of the second fundamental form

$$\tilde{L}_{\alpha\beta} = e^f L_{\alpha\beta} - \frac{\partial}{\partial x^n}(e^f) g_{\alpha\beta}$$

with respect to the conformal change $\tilde{g} = e^{2f}g$ implies that ∂M is totally geodesic. Recall $(x^1, \dots, x^{n-1}, x^n)$ is Fermi coordinates around $P \in \partial M$. By Theorem 5.2 in [LP], there exists a homogeneous polynomial of degree 3, $k_1(x)$ such that the metric $g_2 = e^{2k_1(x)}g_1$ satisfies $\frac{\partial k_1(x)}{\partial x^n}|_{\partial M} = 0$, ∂M being totally geodesic, and

$$R_{ij,k}(P) + R_{jk,i}(P) + R_{ki,j}(P) = 0 \quad \text{for all } 1 \leq i, j, k \leq n.$$

We then get $R_{nn,n}(P) = 0$ and $R_{\alpha\beta,n}(P) + R_{\beta n, \alpha}(P) + R_{n\alpha, \beta}(P) = 0$ for $1 \leq \alpha, \beta \leq n-1$. By the Codazzi equation for $1 \leq \alpha, \beta, \gamma \leq n-1$,

$$R_{\alpha\beta\gamma n} = L_{\alpha\gamma, \beta} - L_{\beta\gamma, \alpha}. \quad (2.6)$$

Differentiating (2.6), we get for $1 \leq \alpha, \beta, \gamma, \delta \leq n-1$

$$R_{\alpha\beta\gamma n, \delta} = L_{\alpha\gamma, \beta\delta} - L_{\beta\gamma, \alpha\delta}.$$

Since ∂M is totally geodesic, after contracting with the metric, we obtain for $1 \leq \alpha, \beta \leq n-1$

$$R_{\alpha n, \beta} = 0 \quad \text{on } \partial M.$$

Hence $R_{\alpha\beta,n}(P) = 0$ for $1 \leq \alpha, \beta \leq n-1$. Let $\bar{g} = g_2 = e^{2k_1(x)}\varphi_1^{\frac{4}{n-2}}g = e^{-2\bar{u}}g$, the metric \bar{g} satisfies all the properties we needed.

□

3 Applications

We denote $[g] = \{\hat{g} \mid \hat{g} = e^{-2u}g\}$ and $[g]_k = \{\hat{g} \mid \hat{g} \in [g] \cap \Gamma_k^+\}$. We call g is k -admissible if and only if $[g]_k \neq \emptyset$. Now the first Yamabe constant on Riemannian manifold (M^n, g) with nonempty boundary ∂M can be defined as ([E1])

$$\mathcal{Y}_1[g] = \inf_{u \in C^1(M), u \neq 0, \int_M u^{\frac{2n}{(n-2)}} = 1} \left(\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) + \frac{n-2}{2} \oint_{\partial M} h_g u^2 \right).$$

We may define the boundary curvature \mathcal{B}^k for the manifold with umbilic boundary and higher order Yamabe constants $\mathcal{Y}_k[g]$ for $2 \leq k < n/2$ as follows (these concepts were defined in [Cn3], the similar higher order Yamabe constants for the manifolds without boundary have appeared in [GLW, S], which are different with the "well-known" definitions, e.g., see [STW]):

$$\mathcal{B}^k = \sum_{i=0}^{k-1} C(n, k, i) \sigma_i(\lambda(g^{-1} A^T)) \tau^{2k-2i-1},$$

and

$$\mathcal{Y}_k[g] = \begin{cases} \inf_{\hat{g} \in [g]_{k-1}; \text{vol}(\hat{g})=1} \mathcal{F}_k & \text{if } [g]_{k-1} \neq \emptyset \\ -\infty & \text{if } [g]_{k-1} = \emptyset \end{cases}$$

where $C(n, k, i) = \frac{(n-i-1)!}{(n-k)!(2k-2i-1)!!}$, $A^T = [A_{\alpha\beta}]$ is the tangential part of the Schouten tensor, τ is a function satisfying $L_{\alpha\beta} = \tau g_{\alpha\beta}$, and

$$\mathcal{F}_k(\hat{g}) = \int_M \sigma_k(\lambda(\hat{g}^{-1} A_{\hat{g}})) + \oint_{\partial M} \mathcal{B}_{\hat{g}}^k.$$

If ∂M is totally geodesic with respect to g , then $\mathcal{B}_g^k = 0$. By Theorem 1 we can get the following Theorems 5 and 6 which can be viewed as a generalization of the corresponding theorems in [Cn3].

Theorem 5. *Let $(M, \partial M, g)$ be a compact manifold of dimension $n \geq 3$ with boundary, ∂M is totally geodesic. Suppose that $g \in [g]_{k-1}$, $2 \leq k < n/2$ and $\mathcal{Y}_1, \mathcal{Y}_k > 0$. Then there exists a metric $\hat{g} \in [g]$ such that $A_{\hat{g}} \in \Gamma_k$ and ∂M is totally geodesic under \hat{g} .*

Proof. Following proof is mainly from [GV1]. Comparing with [S], we may prove the theorem by continuity method. Consider a family of equations involving a parameter t ,

$$\begin{cases} \sigma_k^{1/k}(\lambda(g^{-1} A_{\hat{g}}^t)) = f(x) e^{2u_t} & \text{in } M \\ \frac{\partial u_t}{\partial \nu} = 0 & \text{on } \partial M \end{cases} \quad (3.1)$$

where $\hat{g} = e^{-2u_t}g$, $f(x) > 0$ and $t \leq 1$. Since $g \in [g]_{k-1}$, the scalar curvature $R_g > 0$. Then there exists $a > -\infty$ so that A_g^a is positive definite. For $t \in [a, 1]$, we consider the deformation

$$\begin{cases} \sigma_k^{1/k}(\lambda(g^{-1}A_{u_t}^t)) = f(x)e^{2u_t} & \text{in } M \\ \frac{\partial u_t}{\partial \nu} = 0 & \text{on } \partial M \end{cases} \quad (3.2)$$

where $A_{u_t}^t = A_{\hat{g}}^t$ with $\hat{g} = e^{-2u_t}g$, $f(x) = \sigma_k^{1/k}(\lambda(g^{-1}A_{u_a}^a)) > 0$ and $u_a \equiv 0$ is a solution of (3.2) for $t = a$. Let

$$I = \left\{ t \in [a, t_0] \mid \begin{array}{l} \exists \text{ a solution } u_t \in C^{2,\alpha}(M) \text{ of (3.2) with } A_{\hat{g}_t} \in \Gamma_k \\ \text{and } \partial M \text{ being totally geodesic under } \hat{g}_t \end{array} \right\}.$$

It is easy to prove that the linearized operator $L_t : C^{2,\alpha}(M) \cap \left\{ \frac{\partial u}{\partial \nu} \Big|_{\partial M} = 0 \right\} \rightarrow C^\alpha(M)$ is invertible. This together with the implicit theorem imply that the set I is open.

Theorem 1 implies the C^1 and C^2 estimates of the solution to (3.2) which depend only on the upper bound of u . Since $A^t = A^1 + \frac{1-t}{n-2}\sigma_1(A^1)g$, at the maximal point x_0 of u_t , we have $|\nabla u_t| = 0$ and $\nabla^2 u_t(x_0)$ is negative semi-definite, no matter x_0 being interior or boundary point. Hence,

$$f(x_0)^k e^{2ku(x_0)} = \sigma_k(\lambda(g^{-1}A_{u_t}^t)) \leq \sigma_k\left(\lambda\left(g^{-1}\left(A + \frac{1-t}{n-2}\sigma_1(A)g\right)\right)\right) \leq C,$$

where we use $\sigma_1(A) > 0$ and $a \leq t \leq 1$. We then get the upper bound. By the gradient estimate and the assumption $\mathcal{Y}_1 > 0, \mathcal{Y}_k > 0$, we may easily get the lower bound of u . Therefore we conclude that $I = [a, 1]$. We thus finish the proof. \square

If (M, g) is a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Then by [E1], we may assume that the background metric g is a Yamabe metric with its constant scalar curvature $R > 0$ and the boundary is totally geodesic. Then using the same argument of Theorem 5, we may prove that there exists a metric $\hat{g} \in [g]$ such that $A_{\hat{g}} \in \Gamma_k$ and ∂M is totally geodesic under \hat{g} . By [JLL], we can get the following existence result.

Theorem 6. *Let $(M, \partial M, g)$ be a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Suppose that $2 \leq k < n/2$ and $\mathcal{Y}_1, \mathcal{Y}_k > 0$. Then there exists a metric $\hat{g} \in [g]$ such that $\sigma_k(\lambda(A_{\hat{g}})) = 1$ and ∂M is totally geodesic under \hat{g} .*

4 Proof of Theorems 1 and 2

In this section we first prove Theorem 2 for some general functions $a(x)$ and $b(x)$, and a general 2-symmetric tensor S . After we establish the interior a priori estimates (Theorem 2), we study the boundary estimates for special functions $a(x)$, $b(x)$ and a special 2-symmetric tensor S .

Proof of Theorem 2.

(1) *Case (a).*

Let $K = \Delta u + a|\nabla u|^2$. Note that $\Gamma \subset \Gamma_1^+$, we can immediately get

$$0 \leq \operatorname{tr}(W) = (1 + n \frac{1-t}{n-2})\Delta u + (a + nb)|\nabla u|^2 + \operatorname{tr}S \leq (1 + n \frac{1-t}{n-2})K - n\delta_1|\nabla u|^2 + C.$$

Then

$$(1 + n \frac{1-t}{n-2})K \geq n\delta_1|\nabla u|^2 - C > -C.$$

Hence, K has lower bound. We also have

$$|\nabla u|^2 \leq \frac{(1 + n \frac{1-t}{n-2})K + C}{n\delta_1}. \quad (4.1)$$

Without loss of generality, we may assume $K > 0$. Otherwise, $K \leq 0$. By the above inequality (4.1), we know that $|\nabla u|^2 \leq C$. Then we have the C^1 estimates. Furthermore, we have $|\Delta u| \leq C$. From the condition $\Gamma \subset \Gamma_2^+$, we know that $(\operatorname{tr}W)^2 - |W|^2 = 2\sigma_2(W) > 0$. Therefore $|W| \leq C\operatorname{tr}(W) \leq C$ which implies $|\nabla^2 u| \leq C$. We then get C^2 estimates.

Now by the assumption and (4.1), we have

$$|\nabla u|^2 \leq C(K + 1), \quad (4.2)$$

where C depends on $\|a\|_\infty$ and $\|b\|_\infty$. By (4.2), we can obtain

$$\Delta u = K - a|\nabla u|^2 \leq K + \|a\|_\infty|\nabla u|^2 \leq C(K + 1).$$

By the condition $\Gamma \subset \Gamma_2^+$ again, we know that $|W_{ij}| \leq C\operatorname{tr}(W)$ which implies

$$|\nabla^2 u| \leq C(K + 1), \quad (4.3)$$

where C depends only on $\|a\|_\infty$ and $\|b\|_\infty$ as well. (4.2) and (4.3) are the fundamental inequalities which we will use over and over again.

In order to prove that K is bounded, similar as [Cn2], we consider an auxiliary function $H = \eta K$ in a neighborhood B_r , where $0 \leq \eta \leq 1$ is a cutoff function depending only on r such that $\eta = 1$ in $B_{\frac{r}{2}}$ and $\eta = 0$ outside B_r , $|\nabla \eta| \leq \frac{C\eta^{1/2}}{r}$, $|\nabla^2 \eta| \leq \frac{C}{r^2}$.

We begin to derive the interior C^1 and C^2 estimates.

At the maximum point of H , x_0 , after choosing normal coordinates, we have

$$0 = H_i = \eta_i K + \eta K_i$$

That is

$$K_i = -\frac{\eta_i}{\eta} K.$$

We also have

$$0 \geq H_{ij} = \eta_{ij} K + \eta K_{ij} + \eta_i K_j + \eta_j K_i.$$

Note that $|\nabla \eta| \leq \frac{C\eta^{1/2}}{r}$, $|\nabla^2 \eta| \leq \frac{C}{r^2}$, we have

$$0 \geq H_{ij} = \eta K_{ij} + \Lambda_{ij} K,$$

where

$$\Lambda_{ij} = \eta_{ij} - \frac{2\eta_i \eta_j}{\eta} \geq -C\delta_{ij}$$

and C depends only on r .

Denote

$$P^{ij} = F^{ij} + \frac{1-t}{n-2} (\Sigma_l F^{ll}) \delta^{ij}.$$

Since $t \leq 1$, P^{ij} is still elliptic. By use of Ricci identities, we have

$$|u_{ijkk} - u_{kkij}| \leq C|\nabla^2 u| \leq C(K+1),$$

and

$$|u_{ijk} - u_{kij}| \leq C|\nabla u| \leq C(K^{1/2} + 1).$$

We then have

$$\begin{aligned} 0 &\geq \eta P^{ij} H_{ij} \\ &= \eta^2 P^{ij} K_{ij} + \eta \Lambda_{ij} P^{ij} K \\ &\geq \eta^2 P^{ij} \sum_k [u_{ijkk} + 2a(u_{ki}u_{kj} + u_{ijk}u_k) + a_{ij}u_k^2 + 4a_i u_{kj}u_k] \\ &\quad - C \left(\sum_i F^{ii} \right) (1+K) \\ &\geq \eta^2 P^{ij} \sum_k [u_{ijkk} + 2a(u_{ki}u_{kj} + u_{ijk}u_k)] - C(\sum_i F^{ii})(1 + \eta^{3/2} K^{3/2}). \end{aligned} \quad (4.4)$$

Now we estimate the terms $\sum_k P^{ij} u_{ijkk}$ and $\sum_k P^{ij} (u_{ki}u_{kj} + u_{ijk}u_k)$ respectively.

$$\sum_k P^{ij} u_{ijkk} = F^{ij} \sum_k [W_{ijkk} - 2a(u_{ik}u_{jk} + u_{ikk}u_j) - 2b(u_{lkk}u_l + u_{lk}u_{lk})g_{ij}]$$

$$\begin{aligned}
& -a_{kk}u_iu_j - 4a_ku_{ik}u_j - b_{kk}u_l^2g_{ij} - 4b_ku_{lk}u_lg_{ij} - S_{ijkk}] \\
& \geq \sum_k f_{kk} + F^{ij} \sum_k [-2a(u_{ik}u_{jk} + u_{kki}u_j) - 2b(u_{kkl}u_l + u_{lk}u_{lk})g_{ij}] \\
& \quad - C \sum F^{ii}(1 + K^{3/2}) \\
& \geq \sum_k f_{kk} + F^{ij} \sum_k [-2au_{ik}u_{jk} - 2au_j(-2au_{ki}u_k - a_iu_k^2 - \frac{\eta_i}{\eta}K) \\
& \quad - 2bu_{lk}^2g_{ij} - 2bu_l(-2au_{kl}u_k - a_lu_k^2 - \frac{\eta_l}{\eta}K)g_{ij}] \\
& \quad - C \sum F^{ii}(1 + K^{3/2}).
\end{aligned}$$

We then get

$$\begin{aligned}
\sum_k P^{ij}u_{ijkk} & \geq \sum_k f_{kk} + F^{ij} \sum_k [-2b\Sigma_l(u_{lk})^2g_{ij} - 2au_{ik}u_{jk} + 4a^2u_ju_{ki}u_k \\
& \quad + 4ab\Sigma_l(u_lu_{kl}u_k)g_{ij}] - C\eta^{-1/2} \sum F^{ii}(1 + K^{3/2}). \quad (4.5)
\end{aligned}$$

We also have

$$\begin{aligned}
2aP^{ij}(u_{ijk}u_k + u_{ki}u_{kj}) & = 2aF^{ij}(W_{ijk}u_k - 2au_{ik}u_ju_k - 2bu_{lk}u_lg_{ij}u_k + u_{ik}u_{jk} \\
& \quad + \frac{1-t}{n-2} \sum_{l,k} u_{lk}^2\delta_{ij} - a_ku_iu_ju_k - b_ku_l^2g_{ij}u_k - S_{ijk}u_k) \\
& \geq 2au_kf_k + F^{ij}(-4a^2u_{ik}u_ju_k - 4abu_{lk}u_lu_kg_{ij} + 2au_{ik}u_{jk}) \\
& \quad + \frac{1-t}{n-2}2a \sum_i F^{ii} \sum_{l,k} u_{lk}^2 - C \sum_i F^{ii}(1 + K^{3/2}). \quad (4.6)
\end{aligned}$$

From (4.4), (4.5) and (4.6) we therefore have

$$0 \geq \left(\sum F^{ii} \right) \left(2\left(\frac{1-t}{n-2}a - b\right)\eta^2|\nabla^2u|^2 - C\eta^{3/2}K^{3/2} - C \right). \quad (4.7)$$

Let A be a number such that $A > \sqrt{\frac{2}{\delta_2}}(\frac{1-t}{n-2})$. First, we assume $|\nabla u|^2(x_0) < A|\Delta u|(x_0)$. By $|u_{ij}| \leq C(K+1)$, we know that at the point x_0 , $|u_{ij}| \leq C(|\Delta u| + 1)$. Thus (4.7) becomes

$$0 \geq \sum F^{ii} \left(\frac{2}{n-2} \left(\frac{1-t}{n-2}a - b \right) \eta^2 |\Delta u|^2 - C\eta^{3/2}|\Delta u|^{3/2} - C \right).$$

Hence

$$|\Delta u|(x_0) \leq C$$

and

$$K \leq C.$$

Next we consider the case $|\nabla u|^2(x_0) \geq A|\Delta u|(x_0)$. From $|u_{ij}| \leq C(K+1)$, we know that at the point x_0 , $|u_{ij}| \leq C(|\nabla u|^2 + 1)$. Thus (4.7) becomes

$$0 \geq \sum F^{ii} \left(2\left(\frac{1-t}{n-2}a-b\right)\eta^2 u_{il}^2 - C\eta^{3/2}|\nabla u|^3 - C \right). \quad (4.8)$$

We may assume that W_{ij} is diagonal at the point x_0 ,

$$W_{ii} = u_{ii} + \frac{1-t}{n-2}\Delta u + au_i^2 + b\Sigma_k u_k^2 + S_{ii},$$

and

$$0 = W_{ij} = u_{ij} + au_i u_j + S_{ij}, \quad (i \neq j).$$

Since

$$F^{ii}(u_{il} + \frac{1-t}{n-2}\Delta u g_{il} + S_{il})^2 \leq 2F^{ii}[u_{il}^2 + (\frac{1-t}{n-2}\Delta u g_{il} + S_{il})^2],$$

we obtain

$$\begin{aligned} 2\Sigma_{l,i} F^{ii} u_{li} u_{li} &\geq \Sigma_{l,i} F^{ii} [(u_{il} + \frac{1-t}{n-2}\Delta u g_{il} + S_{il})^2 - 2(\frac{1-t}{n-2}\Delta u g_{il} + S_{il})^2] \\ &\geq \Sigma_i F^{ii} \left[\sum_{j \neq i} (-au_i u_j)^2 + (W_{ii} - au_i^2 - b|\nabla u|^2)^2 \right] \\ &\quad - 2 \left(\frac{1-t}{n-2} \right)^2 \frac{1}{A^2} \Sigma_i F^{ii} |\nabla u|^4 - C \Sigma_i F^{ii} \\ &\geq \Sigma_i F^{ii} [(au_i)^2 |\nabla u|^2 + W_{ii}^2 - 2W_{ii}(au_i^2 + b|\nabla u|^2) + b^2 |\nabla u|^4 \\ &\quad + 2abu_i^2 |\nabla u|^2] - 2 \left(\frac{1-t}{n-2} \right)^2 \frac{1}{A^2} \Sigma_i F^{ii} |\nabla u|^4 - C \Sigma_i F^{ii}. \end{aligned}$$

Since

$$2aF^{ii}W_{ii}u_i^2 \leq F^{ii}(W_{ii}^2 + a^2u_i^4) \leq F^{ii}W_{ii}^2 + a^2F^{ii}u_i^2|\nabla u|^2,$$

we have

$$\begin{aligned} 2\Sigma_{l,i} F^{ii} u_{li} u_{li} &\geq \Sigma_i F^{ii} [(2ab + b^2)u_i^2 + \sum_{j \neq i} b^2 u_j^2] |\nabla u|^2 - 2||b||_\infty f |\nabla u|^2 \\ &\quad - 2 \left(\frac{1-t}{n-2} \right)^2 \frac{1}{A^2} \Sigma_i F^{ii} |\nabla u|^4 - C \Sigma_i F^{ii}. \end{aligned}$$

By the assumption of the theorem case (a), $\min \{2ab + b^2, b^2\} \geq \delta_2 > 0$, and Lemma 1, we then have

$$2\Sigma_{l,i} F^{ii} u_{li} u_{li} \geq \left(\delta_2 - 2 \left(\frac{1-t}{n-2} \right)^2 \frac{1}{A^2} \right) \Sigma_i F^{ii} |\nabla u|^4 - C \Sigma_i F^{ii} |\nabla u|^2 - C \Sigma_i F^{ii}$$

Therefore, by (4.8) we have

$$0 \geq (\Sigma_i F^{ii}) [\delta_1 \left(\delta_2 - 2 \left(\frac{1-t}{n-2} \right)^2 \frac{1}{A^2} \right) \eta^2 |\nabla u|^4 - C\eta^{3/2} |\nabla u|^3 - C\eta |\nabla u|^2 - C].$$

Since $A > 0$ large enough, we have

$$|\nabla u|^2(x_0) \leq C,$$

therefore

$$K \leq C.$$

Proof of Remark. We may estimate the term $2aF^{ii}W_{ii}u_i^2$ as follows. Since $W \in \Gamma_2^+$, we have

$$W_{ii} \leq \text{tr}W = (1 + n \frac{1-t}{n-2})\Delta u + (a + nb)|\nabla u|^2 + \text{tr}S$$

for each i . By use of the condition $|\nabla u|^2(x_0) \geq A\Delta u(x_0)$ for some suitable large number A , we have

$$W_{ii} \leq [(1 + n \frac{1-t}{n-2})\frac{1}{A} + (a + nb)]|\nabla u|^2 + \text{tr}S.$$

Now

$$2aF^{ii}W_{ii}u_i^2 \leq 2\|a\|_\infty F^{ii}u_i^2 \{ [(1 + n \frac{1-t}{n-2})\frac{1}{A} + (a + nb)]|\nabla u|^2 + \text{tr}S \}.$$

Then

$$\begin{aligned} 2\Sigma_{l,i} F^{ii}u_{li}u_{li} &\geq \sum_i F^{ii} [\left(b^2 - a^2 + 2ab - 2n\|a\|_\infty b \right) u_i^2 + \sum_{j \neq i} b^2 u_j^2] |\nabla u|^2 \\ &\quad - 2\|b\|_\infty f |\nabla u|^2 - 2 \left(\frac{1-t}{n-2} \right)^2 \frac{1}{A^2} \sum_i F^{ii} |\nabla u|^4 \\ &\quad - 2\|a\|_\infty (1 + n \frac{1-t}{n-2}) \frac{1}{A} \sum_i F^{ii} u_i^2 |\nabla u|^2 - C \sum_i F^{ii} \end{aligned}$$

Using the condition $\min\{b^2 - a^2 + 2ab - 2n\|a\|_\infty b, b^2\} \geq \delta_2 > 0$ in Remark, we may get

$$\begin{aligned} 2\Sigma_{l,i} F^{ii}u_{li}u_{li} &\geq \left(\delta_2 - 2 \left(\frac{1-t}{n-2} \right)^2 \frac{1}{A^2} - 2\|a\|_\infty (1 + n \frac{1-t}{n-2}) \frac{1}{A} \right) \Sigma_i F^{ii} |\nabla u|^4 \\ &\quad - C\Sigma_i F^{ii} |\nabla u|^2 - C\Sigma_i F^{ii}. \end{aligned}$$

Now substituting this inequality to (4.8) we may get the desire estimate.

(2) Case (b).

Since $a(x) + nb(x) \leq -\delta_3$, by the condition $\Gamma \subset \Gamma_1^+$, we have

$$\begin{aligned} 0 \leq \text{tr}(W) &= (1 + n \frac{1-t}{n-2}) \Delta u + a |\nabla u|^2 + nb |\nabla u|^2 + \text{tr} S \\ &\leq (1 + n \frac{1-t}{n-2}) \Delta u - \delta_3 |\nabla u|^2 + C. \end{aligned}$$

Then

$$|\nabla u|^2 \leq C(\Delta u + 1). \quad (4.9)$$

The proof is similar as the argument in case (a). We take the same auxiliary function $H = \eta(\Delta u + a|\nabla u|^2) \triangleq \eta K$, where $\eta(r)$ is a cutoff function as in case (a). Without loss of generality, we may assume

$$K = \Delta u + a|\nabla u|^2 \gg 1.$$

Since $a(x) \geq 0$, by (4.9), we have

$$\Delta u \leq C(K + 1) \quad (4.10)$$

and

$$|\nabla u|^2 \leq C(K + 1). \quad (4.11)$$

Suppose that the maximum point of H achieves at x_0 , an interior point. Then at this point, we need to note that $|\nabla u|^2$, Δu and K all can be controlled by $C(|\nabla^2 u| + 1)$. By the same computation as in case (a), (4.4), (4.5), (4.6) and (4.7) become

$$0 \geq \eta^2 P^{ij} \sum_k [u_{ijkk} + 2a(u_{ki}u_{kj} + u_{ijk}u_k)] - C \left(\sum_i F^{ii} \right) (1 + |\nabla^2 u|^{3/2}), \quad (4.12)$$

$$\begin{aligned} \sum_k P^{ij} u_{ijkk} &\geq \sum_k f_{kk} + F^{ij} \sum_k [-2b \Sigma_l (u_{lk})^2 g_{ij} - 2a u_{ik} u_{jk} + 4a^2 u_j u_{ki} u_k \\ &\quad + 4ab \Sigma_l (u_l u_{kl} u_k) g_{ij}] - C \eta^{-1/2} \sum_i F^{ii} (1 + |\nabla^2 u|^{3/2}), \end{aligned} \quad (4.13)$$

$$\begin{aligned} 2a P^{ij} (u_{ijk} u_k + u_{ki} u_{kj}) &\geq 2a u_k f_k + F^{ij} (-4a^2 u_{ik} u_j u_k - 4ab u_{lk} u_l u_{lk} g_{ij} + 2a u_{ik} u_{jk}) \\ &\quad + \frac{1-t}{n-2} 2a \sum_i F^{ii} \sum_{l,k} u_{lk}^2 - C \sum_i F^{ii} (1 + |\nabla^2 u|^{3/2}), \end{aligned} \quad (4.14)$$

and

$$0 \geq \left(\sum F^{ii} \right) \left(2 \left(\frac{1-t}{n-2} a - b \right) (\eta |\nabla^2 u|)^2 - C (\eta |\nabla^2 u|)^{3/2} - C \eta |\nabla^2 u| - C \right) \quad (4.15)$$

respectively. (4.15) gives $\eta |\nabla^2 u|(x_0) \leq C$ and hence the bounds of $K, |\nabla^2 u|$ and $|\nabla u|$.

□

Proof of Theorem 1.

Note in this theorem a, b are constants and $S = A$ is the Schouten tensor. Similar with the case (b) in Theorem 2. Let $K = \Delta u + a|\nabla u|^2$. We have (4.9), (4.10) and (4.11). Consider $\bar{H} = \eta K e^{px_n}$ in a neighborhood \bar{B}_r^+ , where $0 \leq \eta \leq 1$ is a cutoff function depending only on r such that $\eta = 1$ in $\bar{B}_{\frac{r}{2}}^+$ and $\eta = 0$ outside \bar{B}_r^+ , and p is a large positive constant.

Step 1. We first prove the maximum point of \bar{H} must be in the interior of M . Assume \bar{H} arrives at its maximum point x_0 on the boundary. By a direct calculation of \bar{H}_n , we can show that $\bar{H}_n|_{x_0} > 0$, which violates the assumption.

Note that η is a function depending only on r , thus at the boundary point we have $\eta_n = 0$. By use of (2.1) and (2.2), we get

$$\begin{aligned} \bar{H}_n|_{x_0} &= \eta e^{px_n} (K_n + pK) \\ &= \eta e^{px_n} (u_{nnn} + u_{\alpha\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) + pK) \\ &= \eta e^{px_n} (u_{nnn} + pK), \end{aligned} \tag{4.16}$$

where the last equality follows from

$$\begin{aligned} &u_{\alpha\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) \\ &= a_n u_\gamma u_\gamma = 0. \end{aligned}$$

Now we need the following Lemma 6:

Lemma 6. *There exists a constant C depending only on a, b, Λ, t and ε , such that at x_0 , $u_{nnn} \geq -C(K + 1)$.*

By Lemma 6 we know that if p is large enough,

$$\begin{aligned} \bar{H}_n|_{x_0} &= \eta e^{px_n} (u_{nnn} + pK) \\ &\geq \eta e^{px_n} ((p - C)K - C) > 0, \end{aligned}$$

which completes the first step of the proof.

Proof of Lemma 6.

By Lemma 5, we may choose a conformal metric $\bar{g} = e^{-2\bar{u}}g$ and $\bar{u}_n|_{\partial M} = 0$ at first. In this metric, ∂M is still totally geodesic and $\bar{A}_{\alpha\beta,n}(x_0) = 0$. We wish to find

a metric $\tilde{g} = e^{-2v} \bar{g}$ such that $u = \bar{u} + v$ is a solution to (1.5). Now

$$\begin{aligned}
W_i^l &= g^{lj} \left(\frac{1-t}{n-2} \Delta u g_{ij} + u_{ij} + a u_i u_j + b |\nabla u|^2 g_{ij} + A_{ij} \right) \\
&= g^{lj} \left((\bar{u} + v)_{ij} + a (\bar{u} + v)_i (\bar{u} + v)_j + b |\nabla (\bar{u} + v)|^2 g_{ij} + A_{ij} \right) \\
&\quad + \frac{1-t}{n-2} \Delta (\bar{u} + v) \delta_i^l \\
&= g^{lj} \left(\bar{A}_{ij} + (a-1) \bar{u}_i \bar{u}_j + \left(b + \frac{1}{2} \right) |\nabla \bar{u}|^2 g_{ij} \right) + \frac{1-t}{n-2} \Delta (\bar{u} + v) \delta_i^l \\
&\quad + g^{lj} (v_{ij} + a \bar{u}_i v_j + a \bar{u}_j v_i + a v_i v_j + b (\bar{u}_k v_l + \bar{u}_l v_k) g^{kl} g_{ij} + b |\nabla v|^2 g_{ij}) \\
&= e^{-2\bar{u}} \bar{g}^{lj} \left(\bar{A}_{ij} + (a-1) \bar{u}_i \bar{u}_j + \left(b + \frac{1}{2} \right) |\bar{\nabla} \bar{u}|_{\bar{g}}^2 \bar{g}_{ij} \right) \\
&\quad + e^{-2\bar{u}} \bar{g}^{lj} (\bar{\nabla}_{ij}^2 v + (\bar{\Gamma}_{ij}^k(\bar{g}) - \Gamma_{ij}^k(g)) v_k + a \bar{u}_i v_j + a \bar{u}_j v_i + a v_i v_j) \\
&\quad + e^{-2\bar{u}} \bar{g}^{lj} (b (\bar{u}_k v_l + \bar{u}_l v_k) \bar{g}^{kl} \bar{g}_{ij} + b |\bar{\nabla} v|_{\bar{g}}^2 \bar{g}_{ij}) \\
&\quad + e^{-2\bar{u}} \frac{1-t}{n-2} (\bar{\Delta}(\bar{u} + v) + \bar{g}^{jk} (\bar{\Gamma}_{ij}^k(\bar{g}) - \Gamma_{ij}^k(g)) (\bar{u}_k + v_k)) \delta_i^l,
\end{aligned}$$

where $\bar{A}_{ij} = \bar{u}_{ij} + \bar{u}_i \bar{u}_j - \frac{1}{2} |\nabla \bar{u}|^2 g_{ij} + A_{ij}$. Then equation (1.5) becomes

$$\begin{cases} F(\bar{g}^{-1} \bar{W}) = e^{2\bar{u}} f(x, \bar{u} + v) & \text{in } \bar{B}_r^+, \\ \frac{\partial v}{\partial x^n} = 0 & \text{on } \Sigma_r, \end{cases} \quad (4.17)$$

where

$$\begin{aligned}
\bar{W}_{ij} &= \bar{A}_{ij} + (a-1) \bar{u}_i \bar{u}_j + \left(b + \frac{1}{2} \right) |\bar{\nabla} \bar{u}|_{\bar{g}}^2 \bar{g}_{ij} \\
&\quad + \bar{\nabla}_{ij}^2 v + (\bar{\Gamma}_{ij}^k(\bar{g}) - \Gamma_{ij}^k(g)) v_k + a \bar{u}_i v_j + a \bar{u}_j v_i + a v_i v_j \\
&\quad + b (\bar{u}_k v_l + \bar{u}_l v_k) \bar{g}^{kl} \bar{g}_{ij} + b |\bar{\nabla} v|_{\bar{g}}^2 \bar{g}_{ij} \\
&\quad + \frac{1-t}{n-2} (\bar{\Delta}(\bar{u} + v) + \bar{g}^{jk} (\bar{\Gamma}_{lk}^p(\bar{g}) - \Gamma_{lk}^p(g)) (\bar{u}_p + v_p)) \bar{g}_{ij}.
\end{aligned}$$

Since $u_n = \bar{u}_n = 0$ on the boundary ∂M , $u_{n\alpha} = \bar{u}_{n\alpha} = 0$. By Lemma 3, we can show $u_{\alpha\beta n} = \bar{u}_{\alpha\beta n} = 0$, therefore $v_n = v_{n\alpha} = v_{\alpha\beta n} = 0$ on ∂M . We then have

$$\bar{W}_{\alpha n}(x_0) = \bar{A}_{\alpha n} + v_{\alpha n} + (\bar{\Gamma}_{\alpha n}^\beta(\bar{g}) - \Gamma_{\alpha n}^\beta(g)) v_\beta = 0.$$

Applying an argument of Lemma 13 in [Cn3], we know $F^{\alpha n}(x_0) = 0$. Now by Lemma 5,

$$\begin{aligned}
\bar{W}_{\alpha\beta n}(x_0) &= (\bar{\Gamma}_{\alpha\beta}^\delta(\bar{g}) - \Gamma_{\alpha\beta}^\delta(g)) v_\delta|_{x_0} \\
&\quad + \frac{1-t}{n-2} (\bar{u}_{nnn} + v_{nnn} + [\bar{g}^{\gamma\delta} (\bar{\Gamma}_{\gamma\delta}^p(\bar{g}) - \Gamma_{\gamma\delta}^p(g)) (\bar{u}_p + v_p)]_{,n}) \bar{g}_{\alpha\beta}.
\end{aligned}$$

Here we have used the fact $\bar{g}_{ij,n} = \bar{g}_{,n}^{ij} = 0$. Use Fermi coordinates, we have on ∂M

$$\frac{\partial g_{\alpha\beta}}{\partial x^n} = \frac{\partial}{\partial x^n} \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right\rangle = \left\langle \nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^\beta} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^\alpha} \right\rangle = -2L_{\alpha\beta},$$

and

$$\begin{aligned} \frac{\partial}{\partial x^n} \Gamma_{\alpha\beta}^\delta(g) &= \frac{1}{2} g^{\delta\gamma} \left(\frac{\partial^2 g_{\gamma\alpha}}{\partial x^\beta \partial x^n} + \frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^n} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^n} \right) \\ &= -g^{\delta\gamma} \left((L_{\gamma\alpha})_\beta + (L_{\gamma\beta})_\alpha - (L_{\alpha\beta})_\gamma \right) \\ &= 0 \end{aligned}$$

where $L_{\alpha\beta}$ is the second fundamental form of the boundary ∂M and $L_{\alpha\beta} = 0$ since ∂M is totally geodesic. In the same way, we have $\frac{\partial}{\partial x^n} \bar{\Gamma}_{\alpha\beta}^\delta(\bar{g}) = 0$ on ∂M . Then $\bar{W}_{\alpha\beta n}(x_0) = \frac{1-t}{n-2} (\bar{u}_{nnn} + v_{nnn}) \bar{g}_{\alpha\beta}$. Similarly, by Lemma 5 we have

$$\bar{W}_{nnn}(x_0) = v_{nnn}(x_0) + \frac{1-t}{n-2} (\bar{u}_{nnn} + v_{nnn})(x_0).$$

Now differentiating (4.17) alone the normal direction and taking its value at x_0 we have

$$\begin{aligned} e^{2\bar{u}(x_0)} f_n(x_0, \bar{u}(x_0) + v(x_0)) &= F^{\alpha\beta} \bar{W}_{\alpha\beta n}(x_0) + F^{nn} \bar{W}_{nnn}(x_0) \\ &= F^{nn} v_{nnn}(x_0) + \frac{1-t}{n-2} (\bar{u}_{nnn} + v_{nnn}) \left(\sum_{i=1}^n F^{ii} \right)(x_0), \end{aligned}$$

where we have used the fact that $F^{\alpha n}(x_0) = 0$. Without loss of generality, one may assume $u_{nnn} = \bar{u}_{nnn} + v_{nnn} \leq 0$. Then by use of the condition (A4) and $|\nabla f| \leq \Lambda f$, we have

$$\begin{aligned} v_{nnn}(x_0) &\geq e^{2\bar{u}(x_0)} \frac{f_n(x_0, \bar{u}(x_0) + v(x_0))}{F^{nn}} \\ &\geq -\frac{\Lambda}{\varepsilon} \sigma_1(\bar{W}) \geq -C(K+1) \end{aligned}$$

where C depends only on the constants Λ, ε , and a, b . Although the covariate derivative is taken with respect to the metric \bar{g} , it is the same if we take the covariate derivative with respect to the metric g on the boundary ∂M , i.e. $v_{nnn}(g) = v_{nnn}(\bar{g})$. Now we have $u_{nnn}(x_0) = \bar{u}_{nnn}(x_0) + v_{nnn}(x_0) \geq -C(K+1)$.

□

Step 2.

By Step 1, we have shown that the maximum point of \overline{H} must be in the interior of M . Then similar with the computation of Theorem 2, we have at the maximam point x_0

$$0 = \overline{H}_i = e^{px_n}(\eta_i K + \eta K_i + p\delta_{in}K\eta)$$

That is

$$K_i = -\left(\frac{\eta_i}{\eta} + p\delta_{in}\right)K.$$

We also have

$$\begin{aligned} 0 \geq \overline{H}_{ij} &= e^{px_n}((\eta_i K + \eta K_i + p\delta_{in}K\eta)p\delta_{jn} + \eta_{ij}K \\ &\quad + \eta K_{ij} + \eta_i K_j + \eta_j K_i + p\delta_{in}K_j\eta + p\delta_{in}K\eta_j). \end{aligned}$$

Note that $|\nabla\eta| \leq \frac{C\eta^{1/2}}{r}$, $|\nabla^2\eta| \leq \frac{C}{r^2}$, we have

$$0 \geq \overline{H}_{ij} = e^{px_n}(\eta K_{ij} + \overline{\Lambda}_{ij}K),$$

where

$$\begin{aligned} \overline{\Lambda}_{ij} &= \eta_{ij} - p\eta_i\delta_{jn} - p\eta_j\delta_{in} - p^2\eta\delta_{in}\delta_{jn} - \frac{2\eta_i\eta_j}{\eta} \\ &\geq -C(p^2 + 1)\delta_{ij} \end{aligned}$$

and C depends only on r . By the above inequalities, similar with (4.4) we have

$$\begin{aligned} 0 &\geq \eta P^{ij}\overline{H}_{ij}e^{-px_n} \\ &= \eta^2 P^{ij}K_{ij} + \eta\overline{\Lambda}_{ij}P^{ij}K \\ &\geq \eta^2 P^{ij} \sum_k [u_{ijkk} + 2a(u_{ki}u_{kj} + u_{ijk}u_k)] - C(\sum_i F^{ii})(1 + |\nabla^2 u|^{3/2}). \end{aligned} \quad (4.18)$$

We estimate the terms $\sum_k P^{ij}u_{ijkk}$ and $\sum_k P^{ij}(u_{ki}u_{kj} + u_{ijk}u_k)$ respectively. As the proof of Theorem 2 (b), we may get (4.13) and (4.14). Then by the cancellations, we may get (4.15). Therefore we get the estimations of $|\nabla^2 u|$ and $|\nabla u|^2$.

□

5 Proof of Theorems 3 and 4

Similar as the proofs of Theorems 1 and 2, we first prove Theorem 4 for some general functions $a(x)$ and $b(x)$, and a general 2-symmetric tensor S . We then study the

boundary estimates for special functions $a(x)$, $b(x)$ and a special 2-symmetric tensor S , i.e. $a(x), b(x)$ are both constants and $S = A$ is the Schouten tensor.

Proof of Theorem 4.

(1) *Case (a).*

Let $H = \eta(\Delta u + a|\nabla u|^2)$ and $K = \Delta u + a|\nabla u|^2$, where $0 \leq \eta \leq 1$ is a cutoff function as before.

Note that $\Gamma \subset \Gamma_1^+$ and $V = \frac{t-1}{n-2}(\Delta u)g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2g + S$, we can immediately get

$$0 \leq \text{tr}(V) = (n\frac{t-1}{n-2} - 1)\Delta u - (a + nb)|\nabla u|^2 + \text{tr}S \leq (n\frac{t-1}{n-2} - 1)K - n\delta_1|\nabla u|^2 + C,$$

Hence, $|\nabla u|^2 \leq \frac{(n\frac{t-1}{n-2} - 1)K + C}{n\delta_1}$. Thus we have

$$|\nabla u|^2 \leq C(K + 1), \quad (5.1)$$

where C depends only on $\|a\|_\infty, \|b\|_\infty$ and δ_1 . By (5.1), we can obtain $\Delta u < C(K + 1)$ and

$$|\nabla^2 u| \leq C(K + 1). \quad (5.2)$$

Let x_0 be an interior point where H achieves its maximum. At x_0 , we have

$$0 = H_i = \eta_i K + \eta K_i,$$

that is

$$K_i = -\frac{\eta_i}{\eta}K.$$

We also have

$$0 \geq H_{ij} = \eta_{ij}K + \eta K_{ij} + \eta_i K_j + \eta_j K_i.$$

Note that $|\nabla \eta| \leq \frac{C\eta^{1/2}}{r}, |\nabla^2 \eta| \leq \frac{C}{r^2}$, we have

$$0 \geq H_{ij} \triangleq \eta K_{ij} + \Lambda_{ij}K,$$

where Λ_{ij} is bounded. If we take

$$Q^{ij} = \frac{t-1}{n-2}(\sum F^{ll})\delta_{ij} - F^{ij},$$

which is also positive definite when $t \geq n - 1$, we can obtain

$$0 \geq \eta Q^{ij}H_{ij} = -\eta F^{ij}H_{ij} + \eta \frac{t-1}{n-2}(\sum F^{ii})H_{kk}.$$

By the same computation as in the case (a) of Theorem 1, we may get

$$0 \geq \sum F^{ii} \left(2 \left(\frac{t-1}{n-2} a + b \right) \eta^2 |\nabla^2 u|^2 - C \eta^{3/2} K^{3/2} - C \eta K - C \right). \quad (5.3)$$

As in the case (a) of Theorem 2, we may discuss (5.3) in two cases. If there exists a constant $A > 0$, such that $|\nabla u|^2(x_0) < A|\Delta u|(x_0)$, we may prove

$$|\Delta u|(x_0) \leq C$$

and

$$K \leq C.$$

Otherwise for any constant $A > 0$ large enough, $|\nabla u|^2(x_0) \geq A|\Delta u|(x_0)$. By use of the assumption that $\min \{2ab + b^2, b^2\} \geq \delta_2 > 0$, we may prove

$$|\nabla u|^2(x_0) \leq C,$$

therefore we have

$$K \leq C.$$

By (5.2), we get the Hessian estimates.

(2) *Case (b).*

We take the same auxiliary function $H = \eta(\Delta u + a|\nabla u|^2) \triangleq \eta K$ as in the case (a), where $0 \leq \eta \leq 1$ is a cutoff function such that $\eta = 1$ in $B_{\frac{r}{2}}$ and $\eta = 0$ outside B_r , and also $|\nabla \eta| \leq \frac{C\eta^{1/2}}{r}$, $|\nabla^2 \eta| \leq \frac{C}{r^2}$.

Since $a(x) + nb(x) \geq \delta_3$, by the condition $\Gamma \subset \Gamma_1^+$ again, we have

$$\begin{aligned} 0 \leq tr(V) &= \left(n \frac{t-1}{n-2} - 1 \right) \Delta u - a |\nabla u|^2 - nb |\nabla u|^2 + trS \\ &\leq \left(n \frac{t-1}{n-2} - 1 \right) \Delta u - \delta_3 |\nabla u|^2 + C, \end{aligned}$$

and then

$$|\nabla u|^2 \leq C(\Delta u + 1). \quad (5.4)$$

Without loss of generality, we may assume

$$K = \Delta u + a |\nabla u|^2 \gg 1.$$

Since $a(x) \geq 0$, by (5.4), we have

$$\Delta u \leq C(K + 1) \quad (5.5)$$

and

$$|\nabla u|^2 \leq C(K+1). \quad (5.6)$$

Suppose that the maximum point of H achieves at x_0 , an interior point, we may get an inequality just replacing K in (5.3) by $|\nabla^2 u|$

$$0 \geq \sum F^{ii} \left(2\left(\frac{t-1}{n-2}a + b\right)\eta^2 |\nabla^2 u|^2 - C(\eta|\nabla^2 u|)^{3/2} - C(\eta|\nabla^2 u|) - C \right). \quad (5.7)$$

The coefficient of the highest order term $\frac{t-1}{n-2}a(x) + b(x) \geq \frac{\delta_3}{n} > 0$ since $a(x) \geq 0$ and $a(x) + nb(x) \geq \delta_3 > 0$. Therefore we can get the bounds of K , $|\nabla^2 u|$ and $|\nabla u|^2$.

□

Proof of Theorem 3.

Note that a, b are two constants, $-S = A$ is the Schouten. Similar as the proof of Theorem 1 Case (b), by (5.5) and (5.6), we only need to estimate $K = \Delta u + a|\nabla u|^2$. Consider $\bar{H} = \eta K e^{px_n}$, where $0 \leq \eta \leq 1$ is a cutoff function as before. We may show the maximum point of \bar{H} must be in the interior of M . Then the argument in Theorem 4 case (b) to get the estimations.

We prove this by contradiction. Assume the maximum point of \bar{H} , x_0 , is on the boundary, then by (2.1), (2.2) and (2.4), we have $u_{\alpha\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) = 0$. Then

$$\bar{H}_n|_{x_0} = \eta e^{px_n} (u_{nnn} + pK).$$

Furthermore, we can get the following Lemma 7 as well:

Lemma 7. *We can find some positive constant C , such that $u_{nnn}(x_0) \geq -C(K+1)$.*

From Lemma 7, we can show that $\bar{H}_n|_{x_0} > 0$ as long as p is large enough, which contradicts with the assumption that x_0 is a maximum point. Hence, \bar{H} achieves its maximum at an interior point.

Proof of Lemma 7.

We may assume $u_{nnn} \leq 0$. Similar as Lemma 6, by Lemma 5, we may choose a conformal metric $\bar{g} = e^{-2\bar{u}}g$ and $\bar{u}_n|_{\partial M} = 0$ at first. In this metric, ∂M is still totally geodesic and $\bar{A}_{\alpha\beta,n}(x_0) = 0$. We wish to find a metric $\tilde{g} = e^{-2v}\bar{g}$ such that $u = \bar{u} + v$ is a solution to (1.7). Then equation (1.7) becomes

$$\begin{cases} F(\bar{g}^{-1}\bar{V}) = e^{2\bar{u}}f(x, \bar{u} + v) & \text{in } \bar{B}_r^+, \\ \frac{\partial v}{\partial x^n} = 0 & \text{on } \Sigma_r, \end{cases} \quad (5.8)$$

where

$$\begin{aligned}
\overline{V}_{ij} &= \frac{t-1}{n-2} (\overline{\Delta}(\overline{u} + v) + \overline{g}^{lk} (\overline{\Gamma}_{lk}^p(\overline{g}) - \Gamma_{lk}^p(g)) (\overline{u}_p + v_p)) \overline{g}_{ij} \\
&\quad - \left(\overline{A}_{ij} + (a-1) \overline{u}_i \overline{u}_j + \left(b + \frac{1}{2} \right) |\overline{\nabla} \overline{u}|_{\overline{g}}^2 \overline{g}_{ij} \right) \\
&\quad - \left(\overline{\nabla}_{ij}^2 v + (\overline{\Gamma}_{ij}^k(\overline{g}) - \Gamma_{ij}^k(g)) v_k + a \overline{u}_i v_j + a \overline{u}_j v_i + a v_i v_j \right) \\
&\quad - \left(b (\overline{u}_k v_l + \overline{u}_l v_k) \overline{g}^{kl} \overline{g}_{ij} + b |\overline{\nabla} v|_{\overline{g}}^2 \overline{g}_{ij} \right).
\end{aligned}$$

Notice that the boundary ∂M preserves totally geodesic, we have $u_n = \overline{u}_n = 0$, and $u_{n\alpha} = \overline{u}_{n\alpha} = 0$. By Lemma 3, we have $u_{\alpha\beta n} = \overline{u}_{\alpha\beta n} = 0$, therefore $v_n = v_{n\alpha} = v_{\alpha\beta n} = 0$ on ∂M . As lemma 6, we have $\overline{V}_{\alpha n}(x_0) = 0$. Employing an argument of Lemma 13 in [Cn3], we know $F^{\alpha n}(x_0) = 0$. By Lemma 5, similar as the computation in the proof of Lemma 6, we have

$$\overline{V}_{\alpha\beta n}(x_0) = \frac{t-1}{n-2} (\overline{u}_{nnn} + v_{nnn}) \overline{g}_{\alpha\beta}$$

and

$$\overline{V}_{nnn}(x_0) = -v_{nnn}(x_0) + \frac{t-1}{n-2} (\overline{u}_{nnn} + v_{nnn})(x_0).$$

Differentiating (5.8) alone the normal direction and taking its value at x_0 we have

$$\begin{aligned}
e^{2\overline{u}(x_0)} f_n(x_0, \overline{u}(x_0) + v(x_0)) &= F^{\alpha\beta} \overline{V}_{\alpha\beta n}(x_0) + F^{nn} \overline{V}_{nnn}(x_0) \\
&= \frac{t-(n-1)}{n-2} F^{nn} v_{nnn}(x_0) + \frac{t-1}{n-2} F^{nn} \overline{u}_{nnn}(x_0) \\
&\quad + \frac{t-1}{n-2} (\overline{u}_{nnn} + v_{nnn}) \left(\sum_{\alpha=1}^{n-1} F^{\alpha\alpha} \right) (x_0).
\end{aligned}$$

Since we have assumed that $u_{nnn}(x_0) \leq 0$, this means that $(\overline{u}_{nnn} + v_{nnn})(x_0) \leq 0$. We therefore have

$$\begin{aligned}
v_{nnn}(x_0) &\geq \frac{n-2}{t-(n-1)} \left[e^{2\overline{u}(x_0)} \frac{f_n(x_0, \overline{u}(x_0) + v(x_0))}{F^{nn}} - \frac{t-1}{n-2} \overline{u}_{nnn}(x_0) \right] \\
&\geq -C(K+1),
\end{aligned}$$

since $t > n-1$, where we have used the condition (A4) and $|\nabla f| \leq \Lambda f$, the constant C depends only on the constants Λ, ε , and a, b, t .

□

Now $\overline{H} = \eta K e^{px_n}$ attains its maximum at an interior point x_0 , we have at x_0

$$0 = \overline{H}_i = e^{px_n}(\eta_i K + \eta K_i + p\delta_{in}K\eta),$$

that is

$$K_i = -\left(\frac{\eta_i}{\eta} + p\delta_{in}\right)K.$$

We also have

$$\begin{aligned} 0 \geq \overline{H}_{ij} &= e^{px_n}((\eta_i K + \eta K_i + p\delta_{in}K\eta)p\delta_{jn} + \eta_{ij}K \\ &\quad + \eta K_{ij} + \eta_i K_j + \eta_j K_i + p\delta_{in}K_j\eta + p\delta_{in}K\eta_j). \end{aligned}$$

Then

$$0 \geq \overline{H}_{ij} \triangleq e^{px_n}(\eta K_{ij} + \overline{\Lambda}_{ij}K),$$

where $\overline{\Lambda}_{ij}$ is bounded. Taking

$$Q^{ij} = \frac{t-1}{n-2}(\sum F^{ll})\delta_{ij} - F^{ij},$$

as the proof of Theorem 4, we can obtain

$$0 \geq \eta Q^{ij} \overline{H}_{ij} e^{-px_n} = -\eta F^{ij} \overline{H}_{ij} e^{-px_n} + \eta \frac{t-1}{n-2} (\sum F^{ii}) \overline{H}_{kk} e^{-px_n}.$$

By the same argument as Case (b) of Theorem 4, we have (5.7). Therefore we get the estimations of $|\nabla^2 u|$ and $|\nabla u|^2$.

□

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Addresses:

Yan He: Centre for Mathematical Sciences, Zhejiang University, Hangzhou 310027, China.
 Weimin Sheng: Department of Mathematics, Zhejiang University, Hangzhou 310027, China; and Centre for Mathematics and its Applications, the Australian National University, Canberra, ACT 0200, Australia.

E-mail: helenaig@hotmail.com, weimins@zju.edu.cn